POSITIVE SOLUTIONS FOR FUNCTIONAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. Sufficient conditions which guarantee the existence of positive concave solutions of a boundary value problem for a second order functional differential equation are provided. The results are obtained by applying the Krasnoselski's fixed point theorem on cones.

1. INTRODUCTION

We investigate when a boundary value problem for a second order functional differential equation admits positive concave solutions.

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^+$ the set of nonnegative reals. For a fixed number $r \in \mathbb{R}^+$ we denote by C_r the Banach space of all continuous functions $\phi: [-r, 0] =: J \to \mathbb{R}$ endowed with the sup-norm $||\phi||_J := sup\{|\phi(s)|: s \in J\}$. In the sequel we shall work especially with the set

$$C_{r,0}^+ := \{\phi \in C_r : \phi(s) \geq 0 = \phi(0), s \in J\}.$$

Also, for any continuous function x defined on the interval $[-r, 1]$ and any $t \in I := [0, 1]$, we shall denote by x_t the element of C_r defined by

$$x_t(s) = x(t+s), \quad s \in J.$$

In this paper we deal with the functional differential equation

$$(p(t)x'(t))' + F(t, x_t) = 0, \quad t \in I, \tag{1.1}$$

where $F : I \times C_r \to \mathbb{R}$ and $p : I \to (0, +\infty)$ are continuous functions. For a good review of this class of functional differential equations we refer to the book by Driver [5], while in the most recent book by Hale and Lunel [9] a detailed exposition of the subject is presented.

We associate equation (1.1) with the following boundary conditions

$$x_0 = \phi, \tag{1.2}$$

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$$ax(1) + p(1)bx'(1) = 0, \tag{1.3}$$

where $\phi \in C^+_{r,0}$ is a given function and a, b are real numbers with $a \geq 0$ and $b > 0$.

Boundary value problems of this kind constitute an interesting subject in the theory of functional differential equations and appeared early in the literature. (See [7, 8].) For the existence of solutions of such boundary value problems, fixed points theorems and especially contraction principle, Schauder's theorem, topological transversality and its consequences, are mainly used. For more details on this research area we refer to the books [6, 10] and the papers [1, 16-18] as well as the references therein. During the last two decades a great interest is observed on existence results for positive solutions of functional boundary value problems. (See [1, 4, 11-14, 19, 20].) This interest comes from situations involving nonlinear elliptic problems in annular regions and covers the point-delayed case, namely the so called retarded case (see [1, 7, 14, 20]), as well as the case of functional differential equations (see [4, 8, 11-13, 9]).

In this paper our purpose is to establish sufficient conditions for the existence of positive concave solutions of the boundary value problem (1.1) – (1.3). Our main results are obtained by using the following well known fixed point theorem due to Krasnoselskii [15].

Theorem 1.1. *Let $\mathcal{B}$ be a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset cl\Omega_1 \subset \Omega_2$ and let*

$$A\colon\ \mathbb{K} \cap (cl\Omega_2 \setminus \Omega_1) \to \mathbb{K}$$

be a completely continuous operator such that either

$$\|Au\| \leq \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_1 \ \ \textit{and}\ \ \|Au\| \geq \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_2,$$

or

$$\|Au\| \geq \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_1 \ \ \textit{and}\ \ \|Au\| \leq \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_2.$$

Then A has a fixed point in $\mathbb{K} \cap (cl\Omega_2 \setminus \Omega_1)$.

The paper is organized as follows: Section 2 is devoted to the assumptions of our setting and to transform the boundary value problem into an integral equation, where the corresponding operator has some properties. We emphasize on these properties as well. In Section 3 we present the main existence results. In Section 4 we shall make a discussion concerning the choice of Theorem 1.1 in solving the boundary value problem (1.1)-(1.3) and in the last section we give a one- parameter application, just to illustrate the results.

2. The problem setting and the assumptions

Let $C_0(I)$ be the space of all continuous functions $x : I \to \mathbb{R}$ with $x(0) = 0$. This is a Banach space when it is furnished with the sup-norm $\|x\|_I := sup\{|x(s)| \colon s \in I\}$.

We set

$$C_0^+(I) := \{x \in C(I, \mathbb{R}) :\ \ x(t) \geq x(0) = 0,\ \ t \in I\}.$$

In the sequel when functions $\phi \in C^+_{r,0}$ and $x \in C_0^+(I)$ are given, we shall denote by $x(\cdot;\phi)$ the function which is equal to x on I and to ϕ on J. Then for each $t \in I$ the function $x_t(s;\phi) = x(t+s;\phi)$, $s \in J$ is a point of the set $C(J, \mathbb{R})$.

By a solution of the boundary value problem (1.1) – (1.3) we mean a function $x \in C_0(I)$ such that the derivatives x' and $(px')'$ exist on I, it satisfies condition (1.3) as well as the identity

$$(p(t)x'(t;\phi))' + F(t, x_t(\cdot;\phi)) = 0, \quad t \in I. \tag{2.1}$$

Searching for the existence of solutions of the boundary value problem (1.1)-(1.3) we shall reformulate it to an appropriate operator equation. To find this operator equation, we assume that x is a solution and, for simplicity, we let

$$z(t) := F(t, x_t(\cdot;\phi)).$$

Then for each $t \in I$ by integration we get $p(t)x'(t) = p(1)x'(1) + \int_t^1 z(s)ds$. Taking into account that $x(0) = \phi(0) = 0$, we obtain

$$\begin{aligned} x(t) &= \int_0^t \left(\frac{p(1)}{p(s)} x'(1) + \frac{1}{p(s)} \int_s^1 z(\theta)d\theta \right) ds \\ &= p(1)x'(1) \int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)} \int_s^1 z(\theta)d\theta ds \end{aligned} \tag{2.2}$$

and so from (1.3) we get

$$\begin{aligned} 0 &= ax(1) + p(1)bx'(1) \\ &= a\left(p(1)x'(1)P(1) + \int_0^1 \frac{1}{p(s)} \int_s^1 z(\theta)d\theta ds \right) + bp(1)x'(1), \end{aligned}$$

where $P(t) := \int_0^t p(s)^{-1}ds, \;\; t \in I$. Hence we have

$$p(1)x'(1) = -ac \int_0^1 \frac{1}{p(s)} \int_s^1 z(\theta)d\theta ds,$$

where $c := (aP(1) + b)^{-1}$. Therefore, from (2.2), we get

$$x(t) = -acP(t) \int_0^1 \frac{y(s)}{p(s)} ds + \int_0^t \frac{y(s)}{p(s)} ds, \tag{2.3}$$

where we have set

$$y(s) := \int_s^1 z(\theta)d\theta.$$

Now, for our convenience, write equation (2.3) in the form

$$\begin{aligned} x(t) &= \int_0^1 \frac{y(s)}{p(s)} (-acP(t)ds) + \int_0^t \frac{y(s)}{p(s)} \chi_{[0,t]}(s)ds \\ &= \int_0^1 \frac{y(s)}{p(s)} \big(- acP(t)ds + \chi_{[0,t]}(s)ds \big) \\ &= \int_0^1 \frac{1}{p(s)} \int_s^1 z(\theta)d\theta d_s\gamma(t,s), \end{aligned}$$

where

$$\gamma(t,s) := -acP(t)s + min\{s,t\}.$$

It is clear that for each $t \in I$ the function $\gamma(t,s)$ is of bounded variation with respect to s. Applying Fubini's theorem we get

$$x(t) = \int_0^1 z(\theta) \int_0^\theta \frac{1}{p(s)} d_s\gamma(t,s)d\theta$$

and, finally,

$$x(t) = \int_0^1 G(t,\theta)z(\theta)d\theta, \ \ t \in I,$$

where

$$G(t,\theta) := \int_0^\theta \frac{1}{p(s)} d_s\gamma(t,s) = \int_0^\theta \frac{1}{p(s)} d_s(-acP(t)s + s\chi_{[0,t]}(s)).$$

Hence

$$G(t,s) = \begin{cases} (1-acP(t))P(s), & s \leq t \\ (1-acP(s))P(t), & s \geq t. \end{cases}$$

This is the Green's function which corresponds to the problem

$$(p(t)x'(t))' = 0, \ \ (1.2), \ \ (1.3).$$

Clearly, it agrees with the Green's function provided in [3].

Next define the operator $A_\phi : C_0^+(I) \to C_0(I)$ by the formula

$$(A_\phi x)(t) := \int_0^1 G(t,s)F(s, x_s(\cdot;\phi)ds, \ \ t \in I. \tag{2.4}$$

and observe that the following result holds:

Lemma 2.1. *Given $\phi \in C_{r,0}^+$, a function $x \in C(I)$ is a solution of the boundary value problem* (1.1) − (1.3) *if and only if x solves the equation*

$$x = A_\phi x, \tag{2.5}$$

where A_ϕ is the operator defined by (2.4)

Proof. We have shown that if x is a solution of the boundary value problem (1.1) − (1.3), then x solves equation (2.5). So, it remains to show that, if x is a solution of equation (2.5), then x is also a solution of the boundary value problem (1.1)-(1.3).

Indeed, if x solves (2.5), then it solves the equivalent equation (2.3). Obviously it holds $x(0) = 0$ and, moreover,

$$p(t)x'(t) = -ac\int_0^1 \frac{y(s)}{p(s)}ds + \int_t^1 F(s, x_s(\cdot;\phi))ds,$$

which implies (2.1). Also, from (2.3) we have

$$x(1) = (-acP(1)+1)\int_0^1 \frac{y(s)}{p(s)}ds \text{ and } p(1)x'(1) = -ac\int_0^1 \frac{y(s)}{p(s)}ds.$$

Therefore

$$\begin{aligned} ax(1) + p(1)bx'(1) &= (-a^2cP(1) + a - abc)\int_0^1 \frac{y(s)}{p(s)}ds \\ &= (1 - c(b + aP(1)))\int_0^1 \frac{y(s)}{p(s)}ds = 0, \end{aligned}$$

which prove that (1.3) is also satisfied. □

To proceed we establish the following conditions:

(H_1) There exist a continuous function $u : I \to (0, +\infty)$ and a real number $M > 0$ such that

$$F(t, \psi) \leq u(t), \ \ t \in I, \ \ \psi \in C^+_{r,0} : \|\psi\| \leq M$$

and

$$\int_0^1 G(s, s)u(s)ds \leq M.$$

(H_2) There exist continuous functions $v : I \to (0, +\infty)$, $\tau : I \to [0, r]$ nondecreasing functions $w_j : \mathbb{R}^+ \to \mathbb{R}^+$, $j = 1, 2$ and real numbers

$$0 < r_1 < r, \ \ 0 < \sigma < min\{r_1, 1\}, \ \ \rho > 0, \ \ 0 < l < \frac{1}{2}$$

satisfying

$$m := \frac{\rho}{l^2} < M,$$

$$l \leq t - \tau(t) \leq 1 - l, \ \ \text{for all } t \in [\sigma, 1]$$

and

$$w_j(\rho) \geq \lambda\rho, \ \ j = 1, 2$$

where

$$\lambda := \frac{1}{l^2 bc \int_\sigma^1 P(s)v(s)ds}$$

and such that

$$F(t, \psi) \geq \begin{cases} v(t)w_1(\psi(-r_1)), & t \in [0, \sigma] \\ v(t)w_2(\psi(-\tau(t))), & t \in [\sigma, 1], \end{cases}$$

for all $\psi \in C^+_{r,0}$.

(H_3) $p : I \to (0, +\infty)$ is a continuous, nonincreasing function such that the function

$$e^{\eta t}p(t), \ \ t \in I,$$

where

$$\eta := \frac{v_0\lambda\rho}{\int_0^1 u(s)ds},$$

is nondecreasing. Here $v_0 := \inf_{t\in I} v(t)$.

The following lemmas are basic tools in the proof of our main results. The proof of the first of them can be found in [2].

Lemma 2.2. *Let $x \in C(I)$ be a nonnegative and concave function. Then for any $t \in I$ we have*

$$x(t) \geq t(1-t)\|x\|_I.$$

Now consider the cone

$$\Phi := \{\phi \in C^+_{r,0} : \phi(t) \geq l^2\|\phi\|, \ t \in [-r_1, -r_1 + \sigma]\}.$$

Lemma 2.3. *Consider the continuous functions p, F satisfying assumptions $(H_1)-(H_3)$. Let $\phi \in \Phi$ with $m \leq \|\phi\|_J \leq M$ and a concave function $x \in C^+_0(I)$ with $m \leq \|x\|_I \leq M$, where m, M are given in assumptions $(H_2), (H_3)$. Then*

(i) for all $t, \theta \in I$ it holds

$$F(t, x_t(\cdot;\phi)) \geq \eta \int_\theta^1 F(s, x_s(\cdot;\phi))ds$$

and

(ii) the function

$$\Psi(t) := \frac{1}{p(t)} \int_t^1 F(s, x_s(\cdot;\phi))ds, \ t \in I$$

is nonincreasing.

Proof. (i) Taking into account assumptions $(H_1) - (H_3)$ and Lemma 2.2, we see that for every $t \in [\sigma, 1]$ it holds

$$\begin{aligned} F(t, x_t(\cdot;\phi)) &\geq v(t)w_1(x(t-\tau(t);\phi)) \geq v_0 w_1((t-\tau(t)))(1-(t-\tau(t))\|x\|_I) \\ &\geq v_0 w_1(l^2 m) = v_0 w_1(\rho) \geq v_0 \lambda \rho. \end{aligned}$$

Also for every $t \in [0, \sigma]$ we have

$$\begin{aligned} F(t, x_t(\cdot;\phi)) &\geq v(t)w_2(x(t-r_1;\phi)) \geq v_0 w_2(\phi(t-r_1)) \\ &\geq v_0 w_2(l^2 m) = v_0 w_2(\rho) \geq v_0 \lambda \rho. \end{aligned}$$

On the other hand for all $\theta \in I$ we have

$$\eta \int_\theta^1 F(s, x_s(\cdot;\phi))ds \leq \eta \int_0^1 F(s, x_s(\cdot,\phi))ds \leq \eta \int_0^1 u(s)ds,$$

which, due to the previous arguments, is equal to

$$v_0 \lambda \rho \leq F(t, x_t(\cdot;\phi)),$$

for all $t \in I$. This proves statement (i).

(ii) Let $t_1, t_2 \in I$ be such that $t_1 > t_2$. We set

$$q(t) := \frac{1}{p(t)}, \ t \in I.$$

By using statement (i) we have

$$
\begin{aligned}
\Psi(t_1)-\Psi(t_2) &= q(t_1)\int_{t_1}^{1}F(s,x_s(\cdot;\phi))ds - q(t_2)\int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds \\
&= q(t_1)\left(\int_{t_1}^{1}F(s,x_s(\cdot;\phi))ds - \int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds\right) \\
&\quad + (q(t_1)-q(t_2))\int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds \\
&= -q(t_1)\int_{t_2}^{t_1}F(s,x_s(\cdot;\phi))ds + (q(t_1)-q(t_2))\int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds \\
&\leq -q(t_1)\int_{t_2}^{t_1}\eta\int_{t_2}^{1}F(s,x_s(\cdot;\phi))dsdt \\
&\quad + (q(t_1)-q(t_2))\int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds \\
&= \Big(-\eta(t_1-t_2)q(t_1)+q(t_1)-q(t_2)\Big)\int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds.
\end{aligned}
$$

By (H_3) the function $q(t)e^{-\eta t}$ is nonincreasing and thus it holds

$$q(t_1)e^{-\eta t_1} \leq q(t_2)e^{-\eta t_2}$$

or, equivalently,

$$\eta(t_1-t_2) \geq ln\frac{q(t_1)}{q(t_2)}.$$

Since for all $z>0$ it holds $-ln\ z-1+z\geq 0$, we get

$$
\begin{aligned}
\Psi(t_1)-\Psi(t_2) &= \Big(-\eta(t_1-t_2)q(t_1)+q(t_1)-q(t_2)\Big)\int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds \\
&\leq -q(t_1)\left(-ln\frac{q(t_2)}{q(t_1)}-1+\frac{q(t_2)}{q(t_1)}\right)\int_{t_2}^{1}F(s,x_s(\cdot;\phi))ds \leq 0.
\end{aligned}
$$

The last integral is nonnegative because of (H_2) and the proof is complete. □

3. Main results

Define the set

$$\mathbb{K} := \{x\in C_0^+(I) : x \text{ is concave}\},$$

which is a cone in $C_0(I)$. We use the symbol $B(0,d)$ to denote the sphere in the Banach space $C_0(I)$ with center 0 and radius d. Then, as usually, $clB(0,d)$ will denote the closure of $B(0,d)$. Before stating our main result we shall prove two basic lemmas.

Lemma 3.1. *Consider continuous functions p,F satisfying assumptions $(H_1)-(H_3)$. Then for each $\phi\in\Phi$ with $m\leq\|\phi\|_J\leq M$, the operator A_ϕ maps the set $\mathbb{K}\cap(clB(0,M)\setminus B(0,m))$ into $\mathbb{K}$ and it is completely continuous, i.e. it maps bounded sets into relatively compact sets.*

Proof. First observe that since $a \geq 0$ and $b > 0$ it holds $1 - acP(t) > 0$, $t \in I$ and therefore we have $G(t,s) > 0$, for all $t, s \in I$. Moreover, by (H_2) it is clear that

$$F(t, x_t(\cdot, \phi) \geq 0,$$

for all $t \in I$ and $x \in \mathbb{K}$. Hence $A_\phi(x) \geq 0$ for every $x \in \mathbb{K}$.

Next we claim that the image $A_\phi x$ is a concave function for every $x \in \mathbb{K} \cap (clB(0,M) \setminus B(0,m))$.

Indeed, let $t_1, t_2 \in I$ be points such that $t_1 > t_2$. Then it is enough to show that

$$D := (A_\phi x)(t_1) - (A_\phi x)(t_2) - (A_\phi x)'(t_2)(t_1 - t_2) \leq 0.$$

To do this, we take into account that $A_\phi x$ is defined by the right side of (2.3). So we have

$$\begin{aligned} D &= (A_\phi x)(t_1) - (A_\phi x)(t_2) - (A_\phi x)'(t_2)(t_1 - t_2) \\ &= -ac \int_0^1 \frac{1}{p(\theta)} \int_\theta^1 F(s, x_s(\cdot;\phi))dsd\theta \int_{t_2}^{t_1} \frac{ds}{p(s)} \\ &\quad + \int_{t_2}^{t_1} \frac{1}{p(\theta)} \int_\theta^1 F(s, x_s(\cdot;\phi))dsd\theta \\ &\quad - \frac{(t_1 - t_2)}{p(t_2)} \Big[-ac \int_0^1 \frac{1}{p(\theta)} \int_\theta^1 F(s, x_s(\cdot;\phi))dsd\theta + \int_{t_2}^1 F(s, x_s(\cdot;\phi))ds \Big] \\ &\leq -ac \int_0^1 \frac{1}{p(\theta)} \int_\theta^1 F(s, x_s(\cdot;\phi))dsd\theta \Big[\int_{t_2}^{t_1} \frac{ds}{p(s)} - \frac{t_1 - t_2}{p(t_2)} \Big] \\ &\quad - \Big[\frac{t_1 - t_2}{p(t_2)} \int_{t_2}^1 F(s, x_s(\cdot;\phi))ds - \int_{t_2}^{t_1} \frac{1}{p(\theta)} \int_\theta^1 F(s, x_s(\cdot;\phi))dsd\theta \Big]. \end{aligned}$$

Since p is a nonincreasing function, we have

$$\int_{t_2}^{t_1} \frac{ds}{p(s)} - \frac{t_1 - t_2}{p(t_2)} \geq 0. \tag{3.1}$$

On the other hand by Lemma 2.3, for any $\theta > t_2$, we have

$$\frac{1}{p(t_2)} \int_{t_2}^1 F(s, x_s(\cdot;\phi))ds \geq \frac{1}{p(\theta)} \int_\theta^1 F(s, x_s(\cdot;\phi))ds.$$

Hence it holds

$$\frac{t_1 - t_2}{p(t_2)} \int_{t_2}^1 F(s, x_s(\cdot;\phi))ds \geq \int_{t_2}^{t_1} \frac{1}{p(\theta)} \int_\theta^1 F(s, x_s(\cdot;\phi))dsd\theta. \tag{3.2}$$

Taking into account (3.1) and (3.2) we conclude that $D \leq 0$, which proves our claim, namely the concavity of the function $A_\phi x$.

Finally, from (H_1) it follows that $F(t, \cdot)$ maps bounded sets into bounded sets. This means that A_ϕ maps bounded sets into relatively compact sets. Then the uniform continuity of the Green's function $G(t,s)$ with respect to t imply that A_ϕ is a completely continuous operator. □

Lemma 3.2. *Assume that assumption (H_2) is satisfied. Then for every $\phi \in \Phi$ and $x \in \mathbb{K}$ with $\|x\|_I = m$ we have $\|A_\phi\|_I \geq \|x\|_I$.*

Proof. From (H_2) and Lemma 2.2 we have

$$\begin{aligned}
\|A_\phi x\|_I \geq (A_\phi x)(1) &\geq \int_\sigma^1 G(1,s)F(s, x_s(\cdot;\phi))ds \\
&\geq \int_\sigma^1 G(1,s)v(s)w_1(x(s-\tau(s);\phi))ds \\
&\geq \int_\sigma^1 G(1,s)v(s)w_1([s-\tau(s)][1-(s-\tau(s))]\|x\|_I)ds \\
&\geq \int_\sigma^1 G(1,s)v(s)w_1(l^2 m)ds = w_1(\rho)\int_0^1 G(1,s)v(s)ds \\
&\geq \lambda\rho \int_\sigma^1 G(1,s)v(s)ds = \frac{\rho}{l^2} = m = \|x\|_I.
\end{aligned}$$

□

Lemma 3.3. *Assume that assumption (H_1) is satisfied. Then for all $\phi \in C^+_{r,0}$ and $x \in \mathbb{K}$, with $\|\phi\|_J \leq M$ and $\|x\|_I = M$, we have $\|A_\phi\|_I \leq \|x\|_I$.*

Proof. First observe that for all $s \in J$ it holds

$$\max_{t\in I} G(t,s) = G(s,s).$$

From (H_1) we have

$$\|A_\phi x\|_I \leq \int_0^1 G(s,s)F(s,x_s(\cdot;\phi))ds \leq \int_0^1 G(s,s)u(s)ds \leq M = \|x\|_I.$$

□

Now by applying Theorem 1.1 we get the main result of this paper:

Theorem 3.4. *Consider the continuous functions p, F satisfying the assumptions $(H_1)-(H_3)$. Then for any $\phi \in \Phi$ with $m \leq \|\phi\|_J \leq M$, the boundary value problem $(1.1)-(1.3)$ admits at least one positive concave solution $x = x(\cdot;\phi)$ such that*

$$m \leq \|x\|_I \leq M.$$

4. Discussion

We want to elaborate a little on the applicability of the Krasnoselskii's theorem 1.1 in connection with the classical Schauder's fixed point theorem.

As we have seen above, when applying Theorem 1.1 one has to demonstrate that the operator A defined by (2.4) maps the closed set $\mathbb{K} \cap (clB^+(0,M) \setminus B^+(0,m))$ into $\mathbb{K}$. And since the set $clB^+(0,M)$ is a convex closed bounded set and A is a completely continuous operator, which by Lemma 3 maps the closed convex set $clB^+(0,M)$ into itself, one can apply the Schauder's fixed point Theorem to conclude the existence of a positive solution x of the operator equation (2.5), which belongs to $clB^+(0,M)$.

Then we have to do two things:

a) Show that x belongs to the cone $\mathbb{K}$.

b) Show that x does not belong to the ball $B(0, m)$.

For the first argument we must show that x is a concave function, which means that the image of any element of the set $clB^+(0, M)$ is a concave function. To do this we actually have to use the method applied in the first part of Lemma 3.1. But then Lemma 2.3 is needed, where, on the other hand, the fact that the domain of A has only concave functions is essential. Hence we have to assume that the solution x is concave!

For the second argument we must show the the norm of the solution is greater than m. But this can only be done by applying the method of Lemma 3.2, where, notice that, the function x is assumed that it is concave.

From the previous fact we conclude that the Schauder's fixed point Theorem is not able to give the result of Theorem 3.4. Therefore under the conditions given in our problem Theorem 1.1 is the appropriate tool to solve the problem.

5. An application

Consider the boundary value problem

$$(e^{-kt}x'(t))' + |x(t-1)|^{1/2} + |x(t - |t - \frac{1}{3}|)|^{1/2} = 0, \quad t \in I, \tag{4.1}$$

$$x_0 = \phi, \quad x(1) + e^{-k}x'(1) = 0, \tag{4.2}$$

where k is a positive real number such that

$$\Lambda(k) := \frac{e^{2k} - 2k^2 - 2k - 1}{e^k - e^{\frac{k}{3}} - \frac{2k}{3}} < \frac{1}{32}$$

and $\phi \in C^+_{1,0}$. The previous inequality can be realized (for approximately $k < 0.01$) since as, it is easily seen, it holds $\lim_{k \to 0} \Lambda(k) = 0$.

Here we have $r = 1$, $p(t) = e^{-kt}$,

$$F(t, \psi) := |\psi(-1)|^{1/2} + |\psi(-|t - \frac{1}{3}|)|^{1/2}.$$

as well as

$$P(t) = \frac{1}{k}(e^{kt} - 1), \quad c = \frac{k}{e^k - 1 + k},$$

and

$$\int_0^1 G(t, t)dt = \frac{e^{2k} - 2k^2 - 2k - 1}{2k^2(e^k - 1 + k)}.$$

Then the corresponding quantities ρ, M and m are

$$\rho := \frac{(e^k - e^{\frac{k}{3}} - \frac{2k}{3})^2}{256k^2(e^k - 1 + k)^2}, \quad M := \frac{(e^{2k} - 2k^2 - 2k - 1)^2}{k^4(e^k - 1 + k)^2}, \quad m := 16\rho.$$

For the function F the corresponding upper and lower boundary functions u, v and $w_i, i = 1, 2$ needed in assumptions $(H_1), (H_2)$ are defined by

$$u(t) = u_0 := 2\sqrt{M}, \;\; v(t) = v_0 := 1, \;\; t \in I,$$

and

$$w_i(\zeta) := \sqrt{\zeta}, \;\; i = 1, 2$$

while the constant λ appeared in (H_2) is given by

$$\lambda := \frac{16k(e^k - 1 + k)}{e^k - e^{\frac{k}{3}} - 2k/3}.$$

Finally, the delay function $\tau(t) := |t - \frac{1}{3}|, \;\; t \in I$ satisfies the relation

$$\frac{1}{4} \leq t - \tau(t) \leq \frac{3}{4}, \;\; t \in [\frac{1}{3}, 1].$$

Now observe that $\rho = \lambda^{-2}$, thus it holds $w_i(\rho) \geq \lambda\rho$, for $i = 1, 2$.

Keeping these statements in mind, it is not hard to see that all conditions (H_1), (H_2), (H_3) with $l = \frac{1}{4}$ and $\sigma = \frac{1}{3}$ are satisfied. Notice that

$$\eta := \frac{\lambda\rho}{2\sqrt{M}} = \frac{k(e^k - e^{\frac{k}{3}} - 2k/3)}{32(e^{2k} - 2k^2 - 2k - 1)} = \frac{1}{32}\frac{k}{\Lambda(k)} > k.$$

Finally, Theorem 3.4 imply that for any function $\phi \in C^+_{1,0}$ such that

$$16\phi(t) \geq \|\phi\|, \;\; t \in [-1, -\frac{2}{3}]$$

and $m \leq \|\phi\|_J \leq M$ the boundary value problem (4.1), (4.2) admits at least one positive concave solution $x = x(\cdot; \phi)$ such that

$$m \leq \|x\|_I \leq M.$$

References

[1] R. P. Agarwal and D. O'Regan, *Some new existence results for differential and integral equations*, Nonlinear Anal. Theory Methods Appl., **29** (1997), 679–692.

[2] R. P. Agarwal and D. O'Regan, *Twin solutions to singular Dirichlet problems*, J. Math. Anal. Appl., **240** (1999), 433–445.

[3] V. Anuradha, D. D. Hai and R. Shivaji, *Existence results for superlinear semipositone BVP'S*, Proc. Am. Math. Soc., **124** (1996), 757–763.

[4] J. M. Davis, K.R. Prasad and W. K.C. Yin, *Nonlinear eigenvalue problems involving two classes for functional differential equations*, Houston J. Math., **26(3)** (2000), 597–608.

[5] R. D. Driver, *Ordinary and delay differential equations*, Springer Verlag, New York, 1976.

[6] L.H. Erbe, Qingai Kong and B.G.Zhang,, *Oscillation Theory for Functional Differential Equations*, Pure and Applied Mathematics, 1994.

[7] L. J. Grimm and K. Schmitt, *Boundary value problems for differential equations with deviating arguments*, Aequationes Math., **4** (1970), 176–190.

[8] G. B. Gustafson and K. Schmitt, *Nonzero solutions of boundary value problems for second order ordinary and delay-differential equations*, J. Differ. Equations, **12** (1972), 129–147.

[9] J. K. Hale and S.M. V. Lunel, *Introduction to functional differential equations*, Springer Verlag, New York, 1993.

[10] J. Henderson, *Boundary Value Problems for Functional Differential Equations*, World Scientific, 1982.

[11] J. Henderson and W. Hudson, *Eigenvalue problems for nonlinear functional differential equations*, Commun. Appl. Nonlinear Anal., **3** (1996), 51–58.

[12] J. Henderson and W. Yin, *Positive solutions and nonlinear eigenvalue problems for functional differential equations*, Appl. Math. Lett., **12** (1999), 63–68.

[13] D. Jang and P. Weng, *Existence of positive solutions for boundary value problems of second order functional differential equations*, Electron. J. Qual. Theory Differ. Equ., **6** (1998), 1–13.

[14] G. L. Karakostas and P. Ch. Tsamatos, *Positive solutions and nonlinear eigenvalue problems for retarded second order differential equations*, Electron. J. Differential Equations **(2002)59** (2002), 1-11.

[15] M. A. Krasnoselskii, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.

[16] S.K. Ntouyas, Y. Sficas and P.Ch. Tsamatos, *An existence principle for boundary value problems for second order functional differential equations*, Nonlinear Anal. Theory Methods Appl., **20** (1993), 195–209..

[17] P. Ch. Tsamatos, *Boundary value problems for functional differential equations with nonlinear boundary conditions*, Arch. Math. Brno., **34** (1998), 257–266.

[18] P. Ch. Tsamatos, *On a boundary value problem for a system for functional differential equations with nonlinear boundary conditions*, Funkc. Ekvacioj, Ser. Int., **42** (1999), 105–114.

[19] P. Weng and D.Jiang, *Existence of positive solutions for a nonlocal boundary value problem of second-order FDE*, Comput. Math. Appl., **37** (1999), 31–9.

[20] P. Weng and Y. Tian, *Existence of positive solutions for singular $(n, n-1)$ conjugate boundary value problem with delay*, Far East J. Math. Sci., **1(3)** (1999), 367-382.

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